Standing waves on a contracting or expanding current

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In a recent work Longuet-Higgins & Stewart (1961) have studied the changes in wavelength and amplitude of progressive waves of constant frequency as they are propagated into regions of surface divergence or convergence. In the work here described the complementary conditions are assumed. Standing waves of uniform wavelength, λ, exist in an area of uniform surface divergence. Changes in amplitude and wavelength are studied. These changes depend on the existence of the radiation stress which was discovered by Longuet-Higgins & Stewart but the physical interpretation of this stress is simpler for standing than for progressive waves. Three different ways of obtaining the same rate of strain in the direction of the current caused the amplitude to vary as $\lambda^{-\frac{1}{4}}$, $\lambda^{-\frac{3}{4}}$ and $\lambda^{-\frac{5}{4}}$, respectively.

Experiments in which free-standing waves were generated in a tank one wavelength wide which was then made narrower verified the conclusion that contraction does not alter the periodic character of the waves, even though the ratio of amplitude to wavelength becomes so great that they can no longer be treated mathematically by the usual linearized approximation. The shape of the profile then appears to agree well with calculations of Penney & Price (1952).

1. Introduction

In a recent paper Longuet-Higgins & Stewart (1961) have investigated the effect of non-uniform currents on short gravity waves. The physical condition assumed to hold was that the waves are generated continuously with a fixed frequency at a fixed point in the current. The frequency observed at all other fixed points is therefore the same as that of the wave generator, but the wave-number varies with position in the current. There is, however, another possible physical wave condition which is worth investigating, namely, the effect of a non-uniform current on a wave train which already exists, or on waves generated by wind action or other cause in the current itself. Most people will have noticed the sudden appearance of smooth areas in the disturbed water downstream of a lock when the sluice gates are opened. These are where rising turbulent currents spread out at the surface with horizontal divergence.

To represent such situations the effect of non-uniform currents with constant horizontal divergence on a train of standing waves whose length is constant in space but variable in time will be discussed. Such a discussion may be expected to be of interest in another connexion. In discussing the energy changes which progressive waves of constant frequency suffer when they enter a non-uniform current, Longuet-Higgins & Stewart (1960, 1961) have shown that it is necessary to take account of an interaction between waves and currents which, in the case of deep water, is equivalent to endowing them with a 'radiation' stress equal to $\frac{1}{2}E_p$, E_p being the wave energy per unit surface area in a progressive wave. The expansion of a progressive wave train as it moves into a region of increasing current velocity gives rise to a transfer of energy from the wave to the current which is identical with that which would occur during expansion at a rate of strain dU/dx under the action of a surface stress $\frac{1}{2}E_p$. Though this stress $S_{px} = \frac{1}{2}E_p$ per unit area must necessarily be present it is rather difficult to understand in detail the mechanism of its action in progressive waves. On the other hand it will be shown that when standing waves are compressed or expanded there exist vertical planes at which there is no horizontal motion due to the waves so that vertical sheets could be inserted there without interfering with the motion. Changes in the energy of waves when they are compressed or expanded could be regarded as being due to the work done by relative motions of the sheets against a 'radiation' stress, S_{sx}, equal to the force acting on a sheet when there are waves only on one side of it. It is therefore possible to calculate S_{sx} by integrating the pressures acting on a vertical sheet at the nodes of a standing wave on a currentless sea.

2. Two-dimensional analysis

A simple irrotational current system in two dimensions which diverges horizontally is that whose velocity potential is $-\frac{1}{2}c(x^2-z^2)$, z being vertical and positive downwards and x horizontal. The streamlines are rectangular hyperbolas. Consideration of Bernoulli's equation reveals that this flow cannot have a flat free-surface. A small correcting term must be added to the velocity potential to enable the stream function $\psi = 0$ to be a surface of constant pressure. Take as the velocity potential and stream functions of the steady flow

$$\phi_0 = -\tfrac{1}{2}c(x^2-z^2)+m(z^3-3x^2z), \quad \psi_0 = cxz+m(3z^2x-x^3), \tag{1}$$

and consider only the part of the field where the slope of the free surface is so small that z/x is small. The equation for the surface, $\psi = 0$, is

$$c\eta_0 = m(x^2-3\eta_0^2),$$

or, if η_0/x is negligible,

$$\eta_0 = mx^2/c.$$

The condition that the pressure there is constant is satisfied if $m = c^3/2g$, so that

$$\eta_0 = c^2x^2/2g \tag{2}$$

and the field in which this approximation is useful is that for which

$$|x| < gc^{-2} \times (\text{maximum allowable value of } d\eta_0/dx).$$

To discuss the effect of this horizontal divergence on a train of standing waves, the time $t = 0$ will be taken as that at which the waves are at their maximum elevation and the initial displacement as

$$\eta-\eta_0 = D\cos kx. \tag{3}$$

Wavelength and amplitude will change with time owing to the divergence c, but the wavelength will remain independent of x so that it is justifiable to assume for η the form

$$\eta = \eta_0 + f(t)\cos kx, \tag{4}$$

where f and k are functions of t only. An appropriate form for the velocity potential is

$$\phi = B\,e^{-kz}\cos kx - \tfrac{1}{2}c(x^2-z^2) + (c^3/2g)\,(z^3-3x^2z), \tag{5}$$

for which the velocity components are

$$u = Bk\,e^{-kz}\sin kx + cx + (3c^3/g)\,xz,$$
$$w = Bk\,e^{-kz}\cos kx - cz - (3c^3/2g)\,(z^2-x^2).$$

B is a function of time which must be determined by the condition that ϕ is compatible with the free surface (3) and the condition that the pressure is constant there. The compatibility condition is

$$\left(\frac{\partial\eta}{\partial t}\right)_x = w - u\left(\frac{\partial\eta}{\partial x}\right)_t. \tag{6}$$

When the second-order terms are neglected the terms which are not time-dependent cancel and

$$w - u(\partial\eta/\partial k)_t = Bk\cos kx - cf\cos kx + cx\,kf\sin kx, \tag{7}$$

while

$$(\partial\eta/\partial t)_x = \dot{f}\cos kx - xf\dot{k}\sin kx. \tag{8}$$

(7) and (8) can only be consistent with (6) if

$$ck = -\dot{k}, \tag{9}$$

so that

$$k = k_0\,e^{-ct}. \tag{10}$$

Since the distance between vertical planes of particles which are perpendicular to x is proportional to e^{ct} in the undisturbed current, (10) shows that in the disturbed flow particles which at any instant are situated in a nodal plane remain in a nodal plane. The compatibility condition is

$$\dot{f} + cf = Bk. \tag{11}$$

The pressure condition at the free surface is

$$\dot{\phi} - \tfrac{1}{2}(u^2+w^2) + g(\eta_0 + f\cos kx) = 0,$$

and, retaining only first-order terms, this reduces to

$$\dot{B}\cos kx - Bx\dot{k}\sin kx - Bxck\sin kx + fg\cos kx = 0. \tag{12}$$

Since $\dot{k} = -ck$, (12) becomes

$$\dot{B} + fg = 0. \tag{13}$$

Eliminating B between (11) and (13),

$$\ddot{f} + 2c\dot{f} + c^2 f + kfg = 0. \tag{14}$$

Writing $ct = \tau - \tau_0$, where $\tau_0 = \ln(c^2/gk_0)$, (14) becomes

$$\frac{d^2f}{d\tau^2} + 2\frac{df}{d\tau} + f + e^{-\tau}f = 0, \tag{15}$$

and writing $e^{-\tau} = \omega$ so that $\omega = (gk_0/c^2)\,e^{-ct}$, (15) becomes

$$\frac{d^2f}{d\omega^2}-\frac{1}{\omega}\frac{df}{d\omega}+\frac{f}{\omega^2}+\frac{f}{\omega} = 0. \tag{16}$$

This equation is a particular case of a more general one which is quoted, for instance, in the text of Jahnke & Emde's Tables (1945); it has a solution

$$f = \omega J_0(2\omega^{\frac{1}{2}}). \tag{17}$$

The equation to the free surface at any time is therefore

$$\eta-\eta_0 = D\,e^{-ct}\frac{J_0(2\sigma_0 c^{-1}\,e^{\frac{1}{2}ct})}{J_0(2\sigma c^{-1})}\cos(k_0\,e^{-ct}x). \tag{18}$$

Here σ_0 is the frequency of waves of length $2\pi/k_0$ and D is the initial amplitude when $t = 0$. It will be noticed that a single curve in which $\omega J_0(2\omega^{\frac{1}{2}})$ is plotted against $-\ln\omega = \tau$ represents the whole range of values of $(\eta-\eta_0)/D$ for all possible initial values of k and any value of c positive or negative. This curve is shown in figure 1. On a diverging current the changing amplitude is represented by a point on the curve which moves from left to right, while that on a converging current (c negative) is represented by a point which moves in the opposite direction.

The availability of the curve of figure 1 for representing the changes in standing waves on a diverging current is limited to the time during which the wavelength is small compared with the dimensions of the area in which the equations are valid approximations; thus if the maximum allowable value of $d\eta_0/dx$ is ϵ, the maximum linear dimension of the wave in which one wavelength covers the whole field is $\lambda = 2g\epsilon/c^2$ so that $2\pi/k$ must be less than $2g\epsilon/c^2$. Since $\omega = gk/c^2 = 2\pi g/c^2\lambda$, the lowest meaningful value of ω in (17) is π/ϵ. Even for a value of ϵ as high as $\frac{1}{10}$, π/ϵ is 10π and the argument of the Bessel function in (17), namely $2\omega^{\frac{1}{2}}$, is about 11. At this value the Bessel function in (17) is close to its asymptotic approximation so that

$$\omega J_0(2\omega^{\frac{1}{2}}) \sim \pi^{-\frac{1}{2}}\omega^{\frac{3}{4}}\cos(2\omega^{\frac{1}{2}}-\tfrac{1}{4}\pi). \tag{19}$$

It appears therefore that a standing wave in an expanding or contracting flow has an amplitude proportional to $k^{\frac{3}{4}}$, i.e.

$$\text{ampl.} \propto \lambda^{-\frac{3}{4}}. \tag{20}$$

This may be compared with an analogous result for progressive waves given in Longuet-Higgins & Stewart (1961), equation (4.9).

3. Radiation stress in standing waves

Equation (10) shows that the wavelengths expand at the same rate as the distance between particles on wave crests, in other words vertical planes through wave crests always contain the same fluid. The rate at which the energy contained in one wavelength, λ, is transferred to the mean flow can therefore be calculated in two ways.

(i) The result (20) that the amplitude is proportional to $k^{\frac{3}{4}}$ makes it possible to calculate the change in energy contained in one wavelength as λ changes and since the only way in which this change can occur is by means of a force acting on the vertical planes at neighbouring crests which are separating at rate $c\lambda$ this force must be equal to (rate of change of energy per wavelength)/λc.

(ii) The second method is to integrate the pressure over vertical planes through the crests of standing waves when there is no diverging current.

The same result is obtained by both methods. The advantage of carrying out both calculations is that a simple conception of the action of radiative stress is obtained. The existence of radiative stress in progressive waves was discovered,

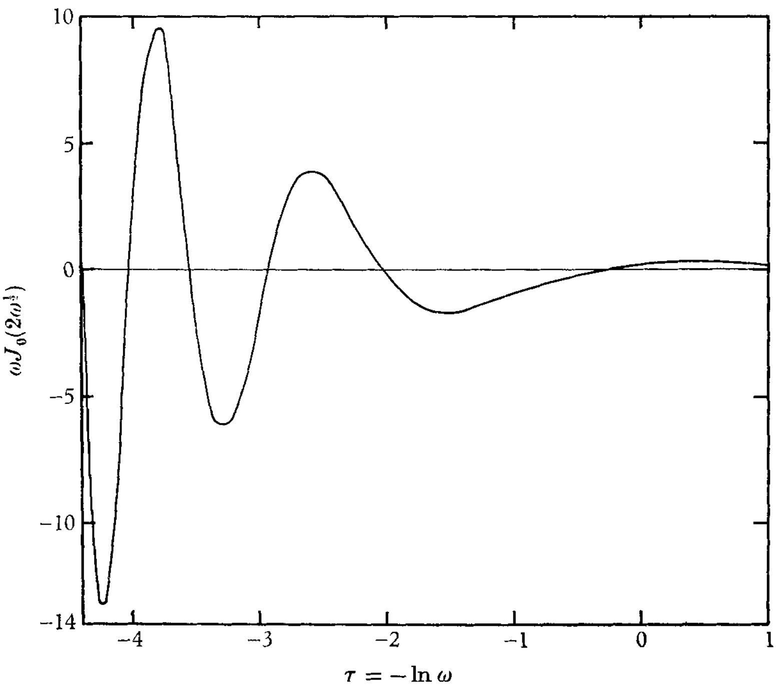

FIGURE 1. Changes in amplitude of standing wave on a contracting or expanding current.

or at any rate first effectively understood by Longuet-Higgins & Stewart (1960, 1961). Their discussion is more general than that of the present work but the conception of radiation stress is simpler for standing than for progressive waves.

The energy of a standing wave of length λ and amplitude a is $E_s\lambda = \frac{1}{4}\rho g a^2 \lambda$. If the wavelength changes from λ to $\lambda + d\lambda$ and the amplitude from a to $a + da$,

$$\frac{dE_s}{E_s} = 2\frac{da}{a} + \frac{d\lambda}{\lambda}. \tag{21}$$

The result of the direct calculation of the relationship between λ and a was that $a \propto \lambda^{-\frac{3}{4}}$ so that

$$da/a = -\tfrac{3}{4}d\lambda/\lambda \quad \text{and} \quad dE_s/E_s = -\tfrac{1}{2}d\lambda/\lambda. \tag{22}$$

The loss in energy per wavelength λdE_s during an extension $d\lambda$ must be due to a radiation stress S_{sx} where $S_{sx}d\lambda = -\lambda dE_s$ and using (22),

$$S_{sx} = \tfrac{1}{2}E_s = \tfrac{1}{8}g\rho a^2. \tag{23}$$

The energy per unit area in a region where a progressive wave is reflected at a vertical plane barrier is twice the energy per unit area of the incident wave; (23) shows that the radiative stress in a standing wave is also twice that of the incident progressive wave.

4. Axisymmetric diverging currents

The smooth mushroom-like areas in the stream below an emptying lock seem to be due to upwelling which spreads out equally in all directions. The same kind of analysis as that used in the two-dimensional case might be used as a model for this situation if the basic current flow is represented by

$$\phi_0 = -\tfrac{1}{2}c(x^2+y^2-2z^2), \tag{24}$$

where y is the co-ordinate parallel to the wave fronts. In order that the free surface may be one of constant pressure, a small corrective term must be added to ϕ_0 which may be taken as $\frac{4}{3}(c^3/g)\{z^3-\frac{3}{2}z(x^2+y^2)\}$, and the free surface is then

$$\eta_0 = \tfrac{1}{2}(c^2/g)(x^2+y^2). \tag{25}$$

A small error will arise with the axisymmetric current which did not affect the two-dimensional case. The correcting term which is necessary to satisfy the pressure condition causes initially straight wave crests to be convected into curves. If this second-order effect can be neglected analysis similar to that for the two-dimensional case can be made. Taking

$$\phi = B\,e^{-kz}\cos kx - \tfrac{1}{2}c(x^2+y^2-2z^2)+\tfrac{4}{3}(c^3/g)\{z^3-\tfrac{3}{2}z(x^2+y^2)\}, \tag{26}$$

and

$$\eta = \eta_0 + f\cos kx, \tag{27}$$

it is found that $ck = -\dot{k}$ as in (9) but the equation of compatibility analogous to (11) is

$$\dot{f}+2cf = Bk$$

and the surface pressure condition is, as before,

$$\dot{B}+fg = 0.$$

Eliminating B,

$$\ddot{f}+3c\dot{f}+2c^2f+kfg = 0. \tag{28}$$

Making the same transformation as before, the equation analogous to (16) is

$$\frac{d^2f}{d\omega^2}-\frac{2}{\omega}\frac{df}{d\omega}+\frac{2f}{\omega^2}+\frac{f}{\omega} = 0;$$

a solution is

$$f = \omega^{\frac{3}{2}}J_1(2\omega^{\frac{1}{2}})$$

so that the asymptotic value is

$$f = \pi^{-\frac{1}{2}}\omega^{\frac{5}{4}}\cos(2\omega^{\frac{1}{2}}-\tfrac{3}{4}\pi). \tag{29}$$

The wave height, a, is therefore proportional to $\lambda^{-\frac{5}{4}}$, or

$$a \propto \lambda^{-\frac{5}{4}}. \tag{30}$$

This result may be used to calculate the radiation stress (if any) acting on vertical planes perpendicular to the wave crests, for, with this type of current, a portion of the surface which was originally square and bounded on two sides by wave crests will remain square so that the energy contained in it is

$$\lambda^2 E_s = \tfrac{1}{4}\lambda^2 g\rho a^2.$$

Expansion from sides of length λ to $\lambda + d\lambda$ changes the energy by an amount $E_s\lambda^2(2da/a + 2d\lambda/\lambda)$. This change must be attributed to stresses S_{sx} and S_{sy} acting on the sides of the square which does work equal to $(S_{sx}+S_{sy})\,\lambda\, d\lambda$ during the expansion. And using (30), $da/a = -\frac{5}{4}d\lambda/\lambda$ so that

$$(S_{sx}+S_{sy})\,\lambda\, d\lambda + E_s\lambda^2(-\tfrac{1}{2}d\lambda/\lambda) = 0, \quad \text{or} \quad S_{sx}+S_{sy} = \tfrac{1}{2}E_s,$$

and since we already know that $S_{xs} = \frac{1}{2}E_s$, this argument shows that $S_{sy} = 0$. This result can also be obtained by integrating pressures over a vertical plane perpendicular to the wave crests when there is no expansion retaining terms of the second order of small quantities.

5. Lateral contraction or expansion without upwelling

If the basic flow is one in which the motion at all depths is confined to planes parallel to $z = 0$ the appropriate choice for ϕ_0 is

$$\phi_0 = -\tfrac{1}{2}c(x^2 - y^2) \tag{31}$$

and the undisturbed free surface is again $\eta_0 = (c^2/2g)\,(x^2+y^2)$. Using the same symbols as in the two previous cases for the wave disturbance, it is found that the equation of compatibility analogous to (11) is $\dot f = Bk$ and the pressure equation (13), namely $\dot B + fg = 0$, is unchanged. Making the same transformations as before the equation for f is

$$\frac{d^2 f}{d\omega^2} + \frac{f}{\omega} = 0 \tag{32}$$

and a solution is

$$f = \omega^{\frac{1}{2}} J_1(2\omega^{\frac{1}{2}}); \tag{33}$$

using the asymptotic approximation the amplitude a is proportional to $\omega^{\frac{1}{4}}$, i.e. to $\lambda^{-\frac{1}{4}}$. This result can be compared with § 5 of Longuet-Higgins & Stewart (1961). It can be verified that it also leads to the conclusion that $S_{sx} = \frac{1}{2}E$ and $S_{sy} = 0$.

6. Effect of currents expanding laterally but not longitudinally

Here the appropriate assumptions for ϕ and η are

$$\phi = B\,e^{-kz}\cos kx - \tfrac{1}{2}c(y^2 - z^2) \quad \text{and} \quad \eta - \eta_0 = f(t)\cos kx \tag{34}$$

and the compatibility condition is

$$\dot f = Bk - cf,$$

while the pressure condition is

$$\dot B + fg = 0,$$

so that

$$\ddot f + c\dot f + kgf = 0$$

and

$$f = D\,e^{-\frac{1}{2}ct}\cos\,(gk - \tfrac{1}{4}c^2)^{\frac{1}{2}}\,t. \tag{35}$$

A rectangle of length λ and initial breadth b has breadth $b\,e^{ct}$ at time t. The energy contained in it is $\lambda b\, e^{ct} E_s$ and since $E_s = \frac{1}{4}\rho g a^2 = \frac{1}{4}\rho g D^2 e^{-ct}$ the energy in the area remains constant. Since only the barriers at right angles to the wave crests move, the stress S_{sy} must therefore be zero.

7. Force exerted by standing waves on a vertical barrier

Waves of small amplitude exert a fluctuating pressure on a vertical wall and the resultant fluctuating force can be divided into two parts: (i) the steady force due to the hydrostatic pressure of fluid whose surface is at the mean level of the standing waves and (ii) the fluctuating part which is the difference between the total force and the steady part defined in (i). If only the first order of small quantities is considered the mean value of the fluctuating part is zero and the force acting on a vertical barrier extending downwards at a wave crest from the surface to such a depth that the fluctuations in pressure on it are negligible is also zero. To calculate the force on a vertical barrier at one side of which there are standing waves, it is necessary to carry the analysis to the second order of small quantities.

The mechanics of standing waves of finite amplitude has been studied by Penney & Price (1951) who showed that standing waves in which all the particles are at rest twice in every period can exist. They showed that there is a single series of such waves depending on the value of a non-dimensional number A and they expressed the velocity potential and displacement of the surface in a Fourier series containing submultiples of the wavelength. The coefficient of each term of this series was a function of time which itself was also expressed as a Fourier series of submultiples of the period. They developed the coefficients of the terms in these functions of time in powers of a single arbitrary number A which determined the amplitude of the waves. The analysis was very complicated and was carried up to terms involving A^5. For the present purpose it is only necessary to include terms up to A^2. Penney & Price showed that there are no terms in ϕ of order A^2. The appropriate expressions for ϕ and η are

$$\phi = \chi(t) + A\sigma k^{-2} e^{-kz} \cos kx \cos \sigma t, \tag{36}$$

$$\eta = Ak^{-1} \cos kx \sin \sigma t - \tfrac{1}{2}A^2 k^{-1} \cos 2kx \sin^2 \sigma t. \tag{37}$$

These equations satisfy the compatibility condition (6) when terms of higher order than A^2 are neglected. The pressure is then given by

$$p/\rho = \dot{\chi}(t) - A\sigma^2 k^{-2} e^{-kz} \cos kx \sin \sigma t - \tfrac{1}{2}A^2 \sigma^2 k^{-2} e^{-2kz} \cos^2 \sigma t + gz. \tag{38}$$

At the surface where $z = \eta$, the condition $p = 0$ is satisfied when terms of higher degree than A^2 are neglected, provided

$$\sigma^2 = gk \quad \text{and} \quad \dot{\chi}(t) = \tfrac{1}{2}A^2 g k^{-1} \cos 2\sigma t. \tag{39}$$

The amplitude, a, of the standing wave is, to the first order, Ak^{-1} so that

$$\rho\dot{\chi}(t) = \tfrac{1}{2} g\rho a^2 k \cos 2\sigma t. \tag{40}$$

This variation in pressure is independent of the depth and is the double-frequency pressure oscillation extending to the bottom of the sea to which several authors have called attention.

The force at any instant exerted by the waves on a vertical barrier of depth D parallel to the wave crests at $x = 0$ is

$$\int_{\eta}^{D} p\,dz - \int_{0}^{D} g\rho z\,dz. \tag{41}$$

The pressure at $x = 0$ is

$$\rho\{\dot{\chi}(t) - a\sigma^2k^{-1}\,e^{-kz}\sin\sigma t - \tfrac{1}{2}a^2\sigma^2\,e^{-2kz}\cos^2\sigma t + gz\}. \tag{42}$$

Integrating (42) the total force to depth D, which is assumed to be much greater than a wavelength, is

$$\rho\{\tfrac{1}{2}Dga^2\cos 2\sigma t - gak^{-1}\sin\sigma t + ga^2(\sin^2\sigma t - \tfrac{1}{4}\cos^2\sigma t - \tfrac{1}{2}\sin^2\sigma t)\}. \tag{43}$$

The mean value of this force is $\frac{1}{8}g\rho a^2$. Comparing this with (23) the mean force of the waves on a vertical wall is equal to $\frac{1}{2}E_s$.

The force per wavelength acting on a vertical barrier placed perpendicular to the wave crests can also be calculated, since no fluid crosses these planes. When this is done it is found that the mean value of the force attributable to the term $-A\sigma^2k^{-2}\,e^{-kz}\cos kx\sin\sigma t$ in (38) is $\frac{1}{4}g\rho a^2\lambda$ while those due to the last two terms $-\frac{1}{2}A^2\sigma^2k^{-2}\,e^{-2kz}\cos^2\sigma t$ and gz in (38) are each $-\frac{1}{8}g\rho a^2\lambda$ so that the total mean force on barriers perpendicular to the wave fronts is zero.

8. Experiments on standing waves

The conclusion reached in § 2, that slow horizontal contraction alters the wavelength of a simple-harmonic wave of small amplitude but preserves its simple-harmonic character, seemed worth verifying experimentally. A tank 102 cm deep, sketched on figure 2, was constructed out of sheet Perspex so that it had two parallel walls and two which converged towards the bottom. The parallel walls were 12 cm apart while the converging walls were 25·4 cm apart at the top and 12 cm at the bottom. In the bottom was a large valve A (figure 2) which could empty the tank to a mark C in about 5 sec when raised by the lever B. The waves were produced by causing a narrow wedge D in the middle of the top of the tank to oscillate vertically.

A 16 mm ciné-camera operated at 69 to 73 frames per second could be placed in two positions so that it could photograph at the top or at levels near the marks C and E.

Since the seatings for the camera were fixed at the same distance from the tank it was possible to take a short length of film covering a few complete periods in the upper position and then move the camera to the lower position and take the surface when it reached the lower position. A disadvantage of this method was that drops were liable to fall off the wave-maker and disturb the wave. This was prevented by installing a trough F (figure 2) which started to travel down a guide as soon as the valve A was raised. It stopped at position G where it could catch the drops. It can be seen at the right-hand side of the top of the photographs (figures 3 and 4 (plates 1 and 2)).

The frames chosen for reproduction in figures 3 and 4 (plates 1 and 2), were those at which the wave crest was at its maximum height during an oscillation.

It was found by counting frames between successive maxima that, within the limits of accuracy obtainable by this method, the period was indistinguishable from $(2\pi l/g)^{\frac{1}{2}}$, the period for small oscillations when the length l was taken as that of the free surface at the time the photograph was taken.

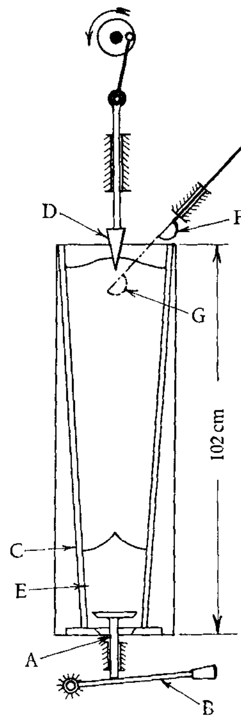

FIGURE 2. Converging tank in which the free-surface length was contracted by lowering the water level.

The theoretical conclusion described by equation (10) that, when a standing wave is compressed between rigid planes at its nodes, its period will alter so that it remains a simple wave is therefore verified, but the decrement due to viscosity was so great that it was impossible to verify the predicted change in amplitude.

Figure 3 (plate 1) shows a case where the initial amplitude is small and the lateral compression of the wave produces a small increase in absolute amplitude in spite of the decrement due to viscosity. The decrement in the 15 or more oscillations which occurred while the water level was falling was never so great that the ratio of wave height to wavelength decreased. It always increased considerably.

If h is the maximum height above the mean level and d the maximum depth below it, the shape of a periodic wave of finite amplitude depends only on $(h+d)/l$. This shape has been calculated approximately by Penney & Price (1952) for a series of values of a non-dimensional number A. For small values of A $(h+d)/l = A/\pi$ and in Penney & Price's approximation the highest wave which has a crest of 90° corresponds with $A = 0{\cdot}592$ and $(h+d)/l = 0{\cdot}128$.

Their expression for the shape of the wave when its crest is at its highest point and the fluid is instantaneously at rest is

$$2\pi y/l = (A+\tfrac{1}{32}A^3-\tfrac{47}{1344}A^5)\cos(2\pi x/l)+(\tfrac{1}{2}A^2-\tfrac{79}{672}A^4)\cos(4\pi x/l)$$
$$+(\tfrac{3}{8}A^3-\tfrac{12563}{59136}A^5)\cos(6\pi x/l)+\tfrac{1}{3}A^4\cos(8\pi x/l)+\tfrac{295}{768}A^5\cos(10\pi x/l). \quad (44)$$

When $A = 0{\cdot}592$ this wave has downward acceleration equal to gravity at its highest point and it should have a pointed crest but the Fourier-series approximation with a finite number of terms cannot represent this. The calculated form for $A = 0{\cdot}592$ is shown in figure 4 (plate 2), and the estimated form at the top of the pointed crest is shown as a broken line (Taylor 1953). The highest value of $(h+d)/l$ observed in the present experiments was that for the wave shown in figure 4. At $l = 14$ cm, the value of $h+d$ was 2·2 cm so that $(h+d)/l = 0{\cdot}16$. To compare this wave with Penney & Price's calculation the profile found from (44) for $A = 0{\cdot}5$ was drawn. This curve is shown at the bottom of figure 4. The corresponding value of $(h+d)/l$ was 0·177. It will be seen that the form of the profiles revealed by the photographs in figure 4 is very similar to that calculated for a wave with nearly the same value of $(h+d)/l$.

The experiment seems to indicate that if the viscous decay had been less, slow compression would permit standing waves to remain periodic even when their amplitude could no longer be regarded as small. It would be interesting to use a larger wave tank to find out whether it is possible to compress waves till a pointed crest is attained.

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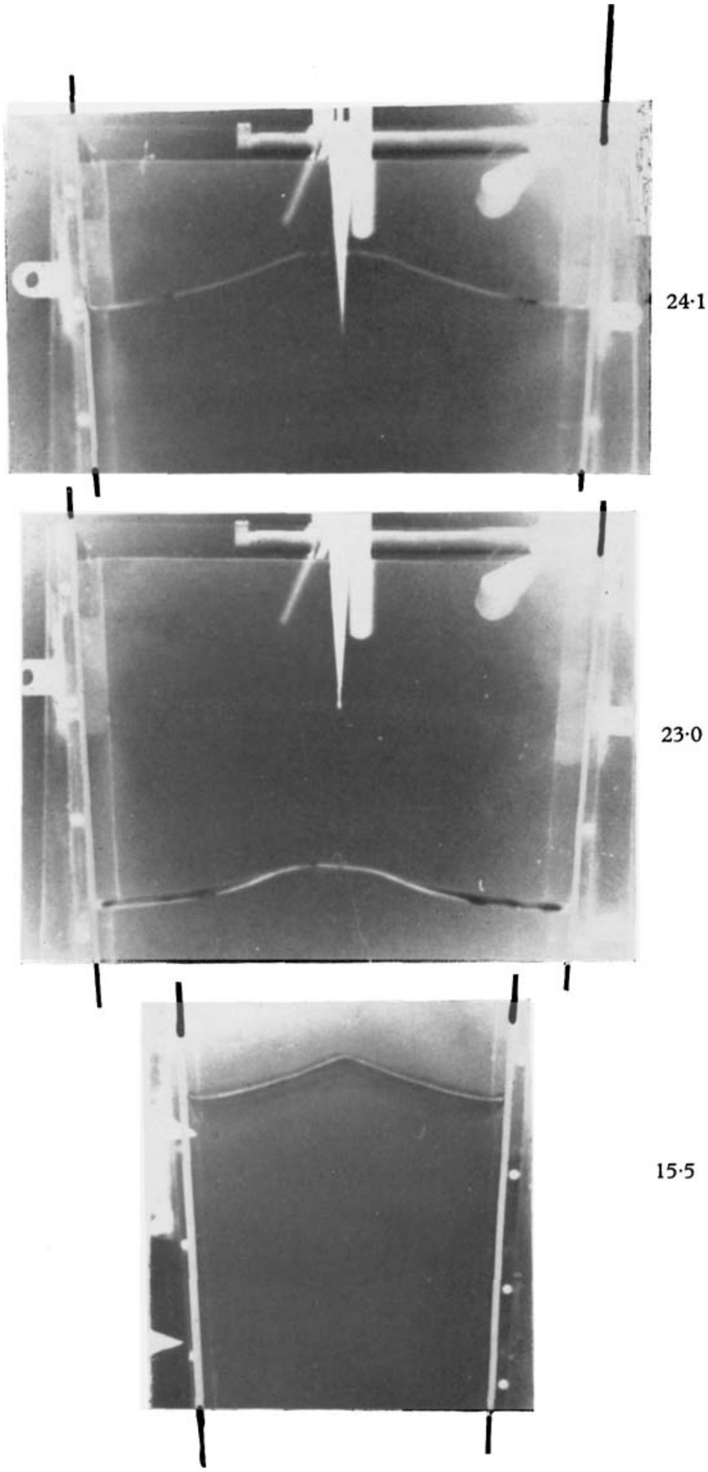

FIGURE 3. Top frame: wave generated at length 24·1 cm; middle frame: wave at maximum amplitude after the length had contracted to 23·0 cm; lower frame: wave at $l = 15{\cdot}5$ cm.

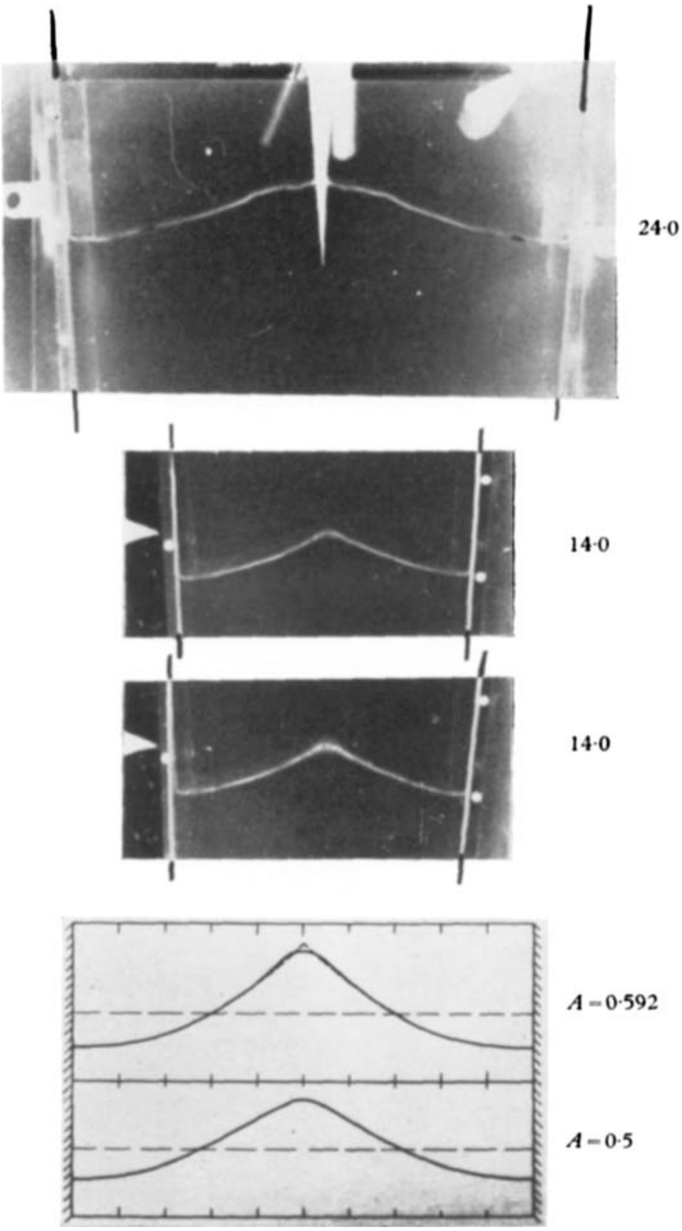

FIGURE 4. Top: wave generated at $l = 24{\cdot}0$ cm. Middle: two consecutive frames when $l = 14{\cdot}0$ cm. Below: calculated wave shapes; the shape for $A = 0{\cdot}50$ can be compared with the photographs above; the crests seem rather sharper but otherwise comparable.